Exercise 1: Show that the second Piola-Kirchhoff stress tensor satisfied the following relations

(a) $\boldsymbol{S} = \boldsymbol{S}^T$ (Symmetry) and (b) $\boldsymbol{S} = \boldsymbol{S}^*$ (Objectivity)

Solution:

(a) The tensor $\boldsymbol{S}$ is related to the Cauchy stresses tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ with

$$\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$

where J is the Jacobian and $\boldsymbol{F}$ is the deformation gradient tensor. Thus,

$$\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} \Rightarrow \boldsymbol{S}^T = \left(J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T}\right)^T = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}^T\boldsymbol{F}^{-T} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \boldsymbol{S} \text{ (due to } \boldsymbol{\sigma} = \boldsymbol{\sigma}^T\text{).}$$

(b) We proceed as follows,

$$\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P} \quad \Rightarrow \boldsymbol{S}^* = \boldsymbol{F}^{*-1}\boldsymbol{P}^*$$
$$\boldsymbol{P}^* = \boldsymbol{Q}\boldsymbol{P} \quad \Rightarrow \boldsymbol{S}^* = \boldsymbol{F}^{*-1}\boldsymbol{Q}\boldsymbol{P}$$
$$\boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F} \quad \Rightarrow \boldsymbol{F}^{*-1} = \boldsymbol{F}^{-1}\boldsymbol{Q}^{-1}$$
$$\Rightarrow \boldsymbol{S}^* = \boldsymbol{F}^{-1}\boldsymbol{Q}^{-1}\boldsymbol{Q}\boldsymbol{P} = \boldsymbol{S}$$

Exercise 2: With respect to then principal axes, the invariants of the Cauchy-Green tensor $\boldsymbol{C}$ are,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \ . \qquad \text{(a)}$$

Here $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the principal stretches. For an isotropic, incompressible material show that,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2}, \quad I_2 = \lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}.$$

Note: Stretch ratio λ, is defined as the ratio of the current length to the initial length of a linear segment after deformation (see kinematics presentation). If we consider a cube of material, in three dimensions, with sides A, B, C at the initial configuration, these sides will become a, b, c after deformation. Thus, a/A = λ_1 , b/B = λ_2 , c/C = λ_3 are the three stretch ratios.

Solution:

From (a) we have for incompressible material $I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 = 1 \Rightarrow \lambda_3^2 = \dfrac{1}{\lambda_1^2\lambda_2^2}$.

We replace the value of λ_3^2 in the first and second equations (a) to obtain the results of the two other invariants of $\boldsymbol{C}$.

Exercise 3: Use the expressions for the invariants in exercise (1) show that,

$$\frac{\partial I_1}{\partial \boldsymbol{C}} = \boldsymbol{I}, \quad \frac{\partial I_2}{\partial \boldsymbol{C}} = I_1 \boldsymbol{I} - \boldsymbol{C}, \quad \frac{\partial \lambda_i^2}{\partial \boldsymbol{C}} = \boldsymbol{A}_i \otimes \boldsymbol{A}_i$$

where $\lambda_i^2 (i = 1,2,3)$ are the principal stretches and $\boldsymbol{A}_i \quad (i = 1,2,3,)$ the principal directions of the deformation tensor $\boldsymbol{C}$.

Hint: use the following relation for the derivative of a scalar function of a tensor variable (B1.144),

$$\frac{\partial \mathcal{W}}{\partial \boldsymbol{T}} = \sum_{i=1}^{3} \frac{\partial \mathcal{W}}{\partial \lambda_i} \boldsymbol{n}_i \otimes \boldsymbol{n}_i \quad \text{where} \quad \mathcal{W}(\boldsymbol{T}) = \phi(\lambda_1, \lambda_2, \lambda_3) \quad .$$

Here $\lambda_i \quad (i = 1,2,3,)$ are the principal principal values of $\boldsymbol{T}$, with the coresponding directions $\boldsymbol{n}_i$.

Solution:

First equality:

$$\begin{aligned}
\frac{\partial I_1}{\partial \boldsymbol{C}} &= \sum_{i=1}^{3} \frac{\partial(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)}{\partial \lambda_i^2} (\boldsymbol{n}_i \otimes \boldsymbol{n}_i) \\
&= \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + \boldsymbol{n}_2 \otimes \boldsymbol{n}_2 + \boldsymbol{n}_3 \otimes \boldsymbol{n}_3 \\
&= \sum_{1}^{3} \boldsymbol{n}_i \otimes \boldsymbol{n}_i \\
&= \boldsymbol{I} \ .
\end{aligned}$$

Second equality:

$$\begin{aligned}
\frac{\partial I_2}{\partial \boldsymbol{C}} &= \sum_{i=1}^{3} \frac{\partial(\lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2)}{\partial \lambda_i^2} (\boldsymbol{n}_i \otimes \boldsymbol{n}_i) \\
&= (\lambda_2^2 + \lambda_3^2)(\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) + (\lambda_3^2 + \lambda_1^2)(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2) + (\lambda_1^2 + \lambda_2^2)(\boldsymbol{n}_3 \otimes \boldsymbol{n}_3) \\
&= (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) + (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2) \\
&+ (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\boldsymbol{n}_3 \otimes \boldsymbol{n}_3) \\
&- \lambda_1^2(\boldsymbol{n}_1 \otimes \boldsymbol{n}_1) - \lambda_2^2(\boldsymbol{n}_2 \otimes \boldsymbol{n}_2) - \lambda_3^2(\boldsymbol{n}_3 \otimes \boldsymbol{n}_3) \\
&= I_1 \boldsymbol{I} - \boldsymbol{C} \ .
\end{aligned}$$

Third equality:

$$\frac{\partial \lambda_i^2}{\partial \boldsymbol{C}} = \sum_{i=1}^{3} \frac{\partial \lambda_i^2}{\partial \lambda_j^2} \boldsymbol{A}_j \otimes \boldsymbol{A}_j = \frac{\partial \lambda_i^2}{\partial \lambda_1^2} \boldsymbol{A}_1 \otimes \boldsymbol{A}_1 + \frac{\partial \lambda_i^2}{\partial \lambda_2^2} \boldsymbol{A}_2 \otimes \boldsymbol{A}_2 + \frac{\partial \lambda_i^2}{\partial \lambda_3^2} \boldsymbol{A}_3 \otimes \boldsymbol{A}_3 \,.$$

In the last sum, only one derivative is non-zero (*i*=1, or 2 or 3) and equal to 1. Thus,

$$\frac{\partial \lambda_i^2}{\partial \boldsymbol{C}} = \boldsymbol{A}_i \otimes \boldsymbol{A}_i$$

Exercise 4: Show that, for an isotropic material, the tensors $\boldsymbol{U}$, $\boldsymbol{C}$ and $\boldsymbol{S}$ have the same principal directions.

Recall that

1: $(\boldsymbol{u} \otimes \boldsymbol{v})(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{v} \cdot \boldsymbol{a})(\boldsymbol{u} \otimes \boldsymbol{b}) = (\boldsymbol{u} \otimes \boldsymbol{b})(\boldsymbol{v} \cdot \boldsymbol{a})$

2: $(\boldsymbol{n}_i \otimes \boldsymbol{n}_i)(\boldsymbol{n}_j \otimes \boldsymbol{n}_j) = \begin{cases} 0 \ \ if \ \ i \neq j \\ (\boldsymbol{n}_i \otimes \boldsymbol{n}_i) \ \ if \ \ i = j \end{cases}.$

Solution

1) Relation (**B**2.108) shows that $\boldsymbol{U}$ has $\boldsymbol{A}_i$ as eigenvectors. By (**B**2.109), one writes the spectral decomposition

$$\boldsymbol{U} = \sum_{i=1}^{3} \lambda_i \boldsymbol{A}_i \otimes \boldsymbol{A}_i \ .$$

With (**B**2.88), one has successively

$$\begin{aligned} \boldsymbol{C} = \boldsymbol{U}\boldsymbol{U} &= \sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_i(\boldsymbol{A}_i \otimes \boldsymbol{A}_i)\lambda_j(\boldsymbol{A}_j \otimes \boldsymbol{A}_j) = \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_i\lambda_j(\boldsymbol{A}_i \otimes \boldsymbol{A}_i)(\boldsymbol{A}_j \otimes \boldsymbol{A}_j) = \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} \lambda_i\lambda_j\delta_{ij}(\boldsymbol{A}_i \otimes \boldsymbol{A}_i) = \\ &= \sum_{i=1}^{3} \lambda_i^2(\boldsymbol{A}_i \otimes \boldsymbol{A}_i) \ . \end{aligned}$$

The isotropic hyperelastic materials have for constitutive equation (**B**6.51)

$$\boldsymbol{S} = 2\frac{\partial \widehat{\mathcal{W}}(\boldsymbol{C})}{\partial \boldsymbol{C}}\,.$$

With (**B**6.64), one obtains relation (**B**6.67) for $\partial \widehat{\mathcal{W}}(\boldsymbol{C})/\partial \boldsymbol{C}$ and finally, one has for (**B**6.68)

$$\boldsymbol{S} = \sum_{i=1}^{3} \frac{1}{\lambda_i}\frac{\partial \phi}{\partial \lambda_i}\,\boldsymbol{A}_i \otimes \boldsymbol{A}_i\,,$$

which shows that $\boldsymbol{S}$ has $\boldsymbol{A}_i$ as eigenvectors.

Exercise 5: Consider the general expression for the energy function,

$$\Phi(I_1, I_2, I_3) = \sum_{i,\,j,\,k=0}^{\infty} C_{ijk}(I_1-3)^i(I_2-3)^j(I_3-1)^k$$

where the invariants are expressed in terms of the principal stretches,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$$

Show that,

(1) in the reference configuration ($\boldsymbol{S} = \boldsymbol{0}$ and $\boldsymbol{C} = \boldsymbol{I}$) we have $C_{000} = 0$.

(2) in the reference configuration, the coefficients C_{100}, C_{010}, C_{001} satisfy the following relation $C_{100} + 2C_{010} + C_{001} = 0$.

Solution:

We expand the given function and retain the three first terms,

$$\begin{aligned}\Phi(I_1, I_2, I_3) \;=\; & C_{000} + C_{100}(I_1-3) + C_{010}(I_2-3) + C_{001}(I_3-1) \\ & + C_{111}(I_1-3)(I_2-3)(I_3-1) + \ldots\ .\end{aligned}$$

In the reference configurations we have,

$\lambda_1 = \lambda_2 = \lambda_3 = 1$ or $\boldsymbol{C} = \boldsymbol{I}$

or

$$I_1 = 3,\ I_2 = 3,\ I_3 = 1$$

Since the energy can be measured from an arbitrary level, we can set this coefficient at the reference configuration equal to zero and thus, $\Phi(3,3,1) = C_{000} = 0$.

Next, calculate the derivatives of the function with respect to the invariants as shown below,

$$\frac{\partial \Phi}{\partial I_1} = C_{100} + C_{111}(I_2 - 3)(I_3 - 1) + \ldots$$

$$\frac{\partial \Phi}{\partial I_2} = C_{010} + C_{111}(I_1 - 3)(I_3 - 1) + \ldots$$

$$\frac{\partial \Phi}{\partial I_3} = C_{001} + C_{111}(I_1 - 3)(I_2 - 3) + \ldots$$

Note here that the first terms in these derivatives are independent of the stretch ratios.

Recalling the following condition that the function should satisfy at the reference configuration,

$$\frac{\partial \Phi}{\partial I_1} + 2\frac{\partial \Phi}{\partial I_2} + \frac{\partial \Phi}{\partial I_3} = 0$$

we obtain,

$$\frac{\partial \Phi}{\partial I_1} + 2\frac{\partial \Phi}{\partial I_2} + \frac{\partial \Phi}{\partial I_3} = C_{100} + 2C_{010} + C_{001} = 0$$

This is true since all other terms in the derivatives are zero at the reference configuration.